JOURNAL OF APPROXIMATION THEORY 57, 169–177 (1989)

Uniqueness in Best One-Sided L_1-Approximation by Algebraic Polynomials on Unbounded Intervals

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Communicated by R. Bojanic

Received February 12, 1987

In this paper we extend the uniqueness result of Bojanic and DeVore [*Enseign. Math.* **12** (1966), 139–164] concerning best one-sided polynomial L_1-approximation on compact intervals to the case of unbounded real intervals. We show that Theorem 3 of the above reference continues to hold in case of noncompact intervals of approximation.

1. INTRODUCTION

In the present literature, one-sided polynomial L_1-approximation has been considered under two different points of view. One aspect is concerned with the question of "goodness of fit" and yields Jackson- or even Bernstein-type results depending on the smoothness of the approximated function and the degree of the polynomials. The first essential result in this direction goes back to Freud [3]; for the case of unbounded intervals which is of interest here, we also mention [4–7, 10, 11]. The other question of interest is connected with the existence, uniqueness, and characterization of best one-sided polynomial L_1-approximations and is related to moment theory and numerical quadrature. The first paper dealing with these questions was written by Bojanic and DeVore [1]; generalizations and extensions of their results concerning more general finite dimensional function spaces instead of polynomials of fixed degree were given in [2, 9, 12, 13]. However, as far as we know uniqueness questions in one-sided L_1-approximation have always been considered with respect to compact intervals of approximation and all proofs of uniqueness implicitly make use of compactness arguments which do not work in case of unbounded intervals. In this paper we therefore want to make a first step to overcome this problem. We will show that Theorem 3 of [1] is valid in case of noncompact intervals, too.

0021-9045/89 $3.00

2. Notation

Let $-\infty \leqslant a < b \leqslant \infty$ be given and let $[a, b]$ be the corresponding real interval with an obvious modification in case of $a = -\infty$ and/or $b = \infty$. For any extended real valued function f defined on $[a, b]$, let $L_n(f)$ be the class of all polynomials p of degree at most n (shortly: $p \in \Pi_n$) satisfying the condition $p(x) \leqslant f(x)$ for all $x \in [a, b]$. Moreover, let μ be an arbitrary nonnegative Borel measure on $[a, b]$ such that all polynomials are integrable with respect to μ on $[a, b]$ and

$$\int_a^b |q(t)| \, d\mu(t) = 0$$

for some polynomial q implies $q = 0$. Now, the class $BL_n(f)$ of best lower L_1-approximations of f with respect to μ is defined to consist of those polynomials $p^* \in L_n(f)$ satisfying

$$\int_a^b p^*(t) \, d\mu(t) = \sup \left\{ \int_a^b p(t) \, d\mu(t) \,\middle|\, p \in L_n(f) \right\}.$$

From now on, we presume that f is integrable over $[a, b]$ with respect to μ, shortly $f \in L_1^\mu[a, b]$. (For the sake of completeness let us note that the existence of a μ-integrable majorant of f on $[a, b]$ would be sufficient to get all results following.) Moreover, we presuppose that f possesses at least one polynomial minorant on $[a, b]$ of degree at most n, i.e., $L_n(f) \neq \varnothing$. Since μ induces a norm on the finite dimensional space Π_n standard arguments yield that $BL_n(f) \neq \varnothing$; i.e., f has at least one best lower L_1-approximation of degree at most n (compare Theorem 1 of [1]). While the existence of a polynomial of best one-sided approximation has been established under very general hypotheses, the example given by Bojanic and DeVore [1] which immediately works in case of unbounded intervals, too, shows that a polynomial of best one-sided approximation is not necessarily unique even for continuous functions which are differentiable on $[a, b]$ with the exception of a finite number of points. In the following we will show that according to the results in the compact case for f being continuous and differentiable on (a, b) we obtain that $BL_n(f)$ consists of precisely one polynomial; in other words, that the best one-sided L_1-approximation problem has a unique solution.

3. The Uniqueness Theorem

In this section we will prove the following theorem which may be interpreted as a generalization of Theorem 3 of [1] by including noncompact intervals.

THEOREM. *Let $n \in \mathbb{N}_0$ and $-\infty \leqslant a < b \leqslant \infty$ be given. Moreover, let μ be a nonnegative Borel measure on $[a, b]$ satisfying the conditions formulated in Section 2 and $f \in L_1^\mu[a, b]$ continuous on $[a, b]$ and differentiable on (a, b) with $L_n(f) \neq \emptyset$. Then $BL_n(f)$ consists of one and only one element.*

To prove the above theorem we need two preparatory lemmas.

LEMMA 1. *Let $n \in \mathbb{N}_0$ be even and $-\infty \leqslant a < b \leqslant \infty$. Moreover, let μ be a nonnegative Borel measure on $[a, b]$ satisfying the conditions formulated in Section 2 and $f \in L_1^\mu[a, b]$ continuous on $[a, b]$ with $L_n(f) \neq \emptyset$. Then for each $p \in BL_n(f)$ the nonnegative function $f - p$ has at least $n/2$ zeros in (a, b).*

Proof. In case $n = 0$ the conclusion is obviously true. So, from now on n may be even and different from zero.

(a) $-\infty < a$ *and* $b < \infty$. Compare [1, Lemmas 3 and 4].

(b) $-\infty = a$ *and* $b = \infty$. Let us assume that $f - p$ has at most $(n/2 - 1)$ real zeros which may be numbered in increasing order: $x_1 < x_2 < \cdots < x_k$ with $k \leqslant n/2 - 1$. If $f - p$ has no zero we always put any of the following factors containing $x_1, ..., x_k$ equal to 1. Now, with the help of the polynomials

$$Q_r(x) := (r + x)(x - x_1)^2 \cdots (x - x_k)^2 (r - x) - \frac{1}{r}, \qquad r \in \mathbb{N},$$

we will show that the above assumption of the number of zeros of $f - p$ will lead to a contradiction to $p \in BL_n(f)$. Obviously, the polynomials Q_r satisfy $Q_r \in \Pi_n$ for all $r \in \mathbb{N}$. Moreover,

$$\int_{-\infty}^{\infty} Q_r(t)\, d\mu(t) = r^2 \int_{-\infty}^{\infty} (t - x_1)^2 \cdots (t - x_k)^2\, d\mu(t) - \int_{-\infty}^{\infty} \left(t^2 (t - x_1)^2 \cdots (t - x_k)^2 + \frac{1}{r} \right) d\mu(t)$$

tends to infinity for $r \to \infty$. Therefore, we may choose $R \in \mathbb{N}$ such that we have simultaneously

$$\int_{-\infty}^{\infty} Q_R(t)\, d\mu(t) > 0 \qquad \text{and} \qquad -R < x_1 < \cdots < x_k < R.$$

By the construction, the set $K := \{x \in \mathbb{R} \mid Q_R(x) \geqslant 0\} \subset [-R, R]$ is compact and not empty. Since

$$Q_R(x_1) = \cdots = Q_R(x_k) = -\frac{1}{R}$$

we have $x_1, \ldots, x_k \notin K$. So, by the continuity of $f-p$ we can find a constant $d>0$ such that

$$f(x)-p(x) \geqslant d \qquad \text{for all} \quad x \in K.$$

Let $\|Q_R\|_\infty^{[-R,R]}$ denote the maximum norm of Q_R with respect to $[-R, R]$. Define

$$p^*(x) := p(x) + d(\|Q_R\|_\infty^{[-R,R]})^{-1} Q_R(x).$$

p^* is a polynomial of degree at most n satisfying

$$f(x)-p^*(x) \geqslant \begin{cases} f(x)-p(x)-d \geqslant 0, & x \in K \\ f(x)-p(x) \geqslant 0, & x \in \mathbb{R}\backslash K \end{cases}$$

and

$$\int_{-\infty}^{\infty} p^*(t)\, d\mu(t) > \int_{-\infty}^{\infty} p(t)\, d\mu(t).$$

This gives the desired contradiction to $p \in BL_n(f)$.

(c) $-\infty = a$ *and* $b < \infty$. As above using the polynomials

$$Q_r(x) := (r+x)(x-x_1)^2 \cdots (x-x_k)^2\, (b-x) - \frac{1}{r}, \qquad r \in \mathbb{N}.$$

(d) $-\infty < a$ *and* $b = \infty$. As above using the polynomials

$$Q_r(x) := (x-a)(x-x_1)^2 \cdots (x-x_k)^2\, (r-x) - \frac{1}{r}, \qquad r \in \mathbb{N}. \quad \blacksquare$$

LEMMA 2. *Let* $n \in \mathbb{N}$ *be odd and* $-\infty \leqslant a < b \leqslant \infty$. *Moreover, let* μ *be a nonnegative Borel measure on* $[a, b]$ *satisfying the conditions formulated in Section* 2 *and* $f \in L_1^\mu[a, b]$ *continuous on* $[a, b]$ *with* $L_n(f) \neq \emptyset$. *Then for each* $p \in BL_n(f)$ *the nonnegative function* $f-p$ *has at least*

$$\frac{n+1}{2} \text{ zeros in } [a, b] \qquad \text{if} \quad -\infty < a \qquad \text{or} \qquad b < \infty,$$

$$\frac{n-1}{2} \text{ zeros on } \mathbb{R} \qquad \text{if} \quad -\infty = a \qquad \text{and} \qquad b = \infty.$$

Proof. (a) $-\infty < a$ *and* $b < \infty$. Compare [1, Lemma 3].

(b) $-\infty = a$ *and* $b = \infty$. Since in case $n = 1$ the conclusion is obviously true we may assume $n \geqslant 3$.

Now, the proof is the same as in Lemma 1 using the polynomials

$$Q_r(x) := (r+x)(x-x_1)^2 \cdots (x-x_k)^2 (r-x) - \frac{1}{r}, \qquad r \in \mathbb{N}.$$

Note that $k \leqslant (n-1)/2 - 1$ implies $Q_r \in \Pi_{n-1}$; i.e., in this case the maximal possible degree of Q_r cannot be attained. This fact causes the difficulties when proving the uniqueness theorem for $\mathbb{R}$ in case of n odd.

(c) $-\infty = a$ *and* $b < \infty$. As in Lemma 1 using the polynomials

$$Q_r(x) := (r+x)(x-x_1)^2 \cdots (x-x_k)^2 - \frac{1}{r}, \qquad r \in \mathbb{N}.$$

(d) $-\infty < a$ *and* $b = \infty$. As in Lemma 1 using the polynomials

$$Q_r(x) := (x-x_1)^2 \cdots (x-x_k)^2 (r-x) - \frac{1}{r}, \qquad r \in \mathbb{N}. \quad \blacksquare$$

Proof of the Theorem. First of all, by Theorem 3 of [1] the case $-\infty < a$ and $b < \infty$ is settled.

Let us now assume that there exist two polynomials p_1 and p_2 satisfying $p_1, p_2 \in BL_n(f)$. It can be easily shown that $p := \frac{1}{2}(p_1 + p_2)$ also lies in $BL_n(f)$ and that $f(z) - p(z) = 0$ for some $z \in [a, b]$ implies $p_1(z) = p_2(z) = f(z)$. Moreover, if $f(z) - p(z) = 0$ and $z \in (a, b)$ we also have $p_1'(z) = p_2'(z) = f'(z)$. For a proof of these facts we refer the reader to [1].

(1) $n \in \mathbb{N}_0$ *and* n *even.* By means of Lemma 1 the nonnegative function $f - p$ has at least $n/2$ zeros in (a, b) which may be numbered in increasing order $a < x_1 < \cdots < x_{n/2} < b$. This implies that $p_1(x_i) = p_2(x_i)$ and $p_1'(x_i) = p_2'(x_i)$, $1 \leqslant i \leqslant n/2$, i.e., that there exists a real constant M satisfying

$$p_1(x) - p_2(x) = M(x-x_1)^2 \cdots (x-x_{n/2})^2.$$

Since $p_1, p_2 \in BL_n(f)$ we also have

$$\int_a^b p_1(t)\, d\mu(t) = \int_a^b p_2(t)\, d\mu(t)$$

which gives

$$0 = M \cdot \int_a^b (t-x_1)^2 \cdots (t-x_{n/2})^2\, d\mu(t).$$

Therefore, $M = 0$ and $p_1 = p_2$.

(2) *$n \in \mathbb{N}$ and n odd.*

(2.1) *$-\infty = a$ and $b = \infty$.* By means of Lemma 2 the nonnegative function $f - p$ has at least $(n-1)/2$ real zeros, $x_1 < \cdots < x_{(n-1)/2}$. This implies $p_1(x_i) = p_2(x_i)$ and $p_1'(x_i) = p_2'(x_i)$, $1 \leqslant i \leqslant (n-1)/2$.

If p_1 and p_2 have the same leading coefficient then $p_1 - p_2 \in \Pi_{n-1}$ and there exists a constant $M \in \mathbb{R}$ satisfying

$$p_1(x) - p_2(x) = M(x - x_1)^2 \cdots (x - x_{(n-1)/2})^2.$$

As in (1) this implies $M = 0$ and $p_1 = p_2$.

On the other hand, if $p_1 - p_2 \in \Pi_n \backslash \Pi_{n-1}$ then there exists a point $z \in \mathbb{R}$ (which may be equal to one of the x_i) and a constant $M \in \mathbb{R}$ satisfying

$$p_1(x) - p_2(x) = M(x - x_1)^2 \cdots (x - x_{(n-1)/2})^2 (x - z).$$

Without loss of generality we may have $M \geqslant 0$. Since we want to show that $M = 0$ we assume the contrary: $M > 0$. Now, we define

$$Q_r(x) := (x - x_1)^2 \cdots (x - x_{(n-1)/2})^2 (x + r) - \frac{1}{r}, \qquad r \in \mathbb{N}.$$

Obviously, we have $Q_r \in \Pi_n$ for all $r \in \mathbb{N}$. Moreover,

$$\int_{-\infty}^{\infty} Q_r(t)\, d\mu(t) = r \int_{-\infty}^{\infty} (t - x_1)^2 \cdots (t - x_{(n-1)/2})^2\, d\mu(t) + \int_{-\infty}^{\infty} \left(t(t - x_1)^2 \cdots (t - x_{(n-1)/2})^2 - \frac{1}{r} \right) d\mu(t)$$

tends to infinity for $r \to \infty$. Therefore, we may choose $R \in \mathbb{N}$ such that we have simultaneously

$$\int_{-\infty}^{\infty} Q_R(t)\, d\mu(t) > 0$$

and $R > \max\{|x_1| + 1, ..., |x_{(n-1)/2}| + 1, |z| + 1\}$. The last requirement implies that $Q_R(x) < 0$ for $x \leqslant -R$ and that $Q_R(x) \geqslant 1$ for $x > R$. By the above construction the compact set $K := \{x \in [-R, R] \mid Q_R(x) \geqslant 0\}$ is not empty and does not contain any of the points $x_1, ..., x_{(n-1)/2}$ since

$$Q_R(x_1) = \cdots = Q_R(x_{(n-1)/2}) = -\frac{1}{R}.$$

So, by the continuity of $f-p$ there exists a constant $d>0$ such that

$$f(x)-p(x)\geqslant d \qquad \text{for all } x\in K.$$

Again, let $\|Q_R\|_\infty^{[-R,R]}$ denote the maximum norm of Q_R with respect to $[-R, R]$ and define

$$\tau_1 := d(\|Q_R\|_\infty^{[-R,R]})^{-1}.$$

Now, we show that there exists a constant $\tau_2>0$ satisfying

$$\frac{M}{2}(x-x_1)^2\cdots(x-x_{(n-1)/2})^2(x-z)\geqslant \tau_2 Q_R(x)$$

for all $x\in(R,\infty)$.

Obviously, it is sufficient to prove that there exists $\tau_2>0$ satisfying

$$\frac{M}{2}(x-z)\geqslant \tau_2(x+R), \qquad x\in(R,\infty),$$

i.e.,

$$\tau_2\leqslant\frac{M(x-z)}{2(x+R)}, \qquad x\in(R,\infty).$$

Since

$$\lim_{x\to\infty}\frac{M(x-z)}{2(x+R)}=\frac{M}{2}>0$$

there exists $x_0>0$ satisfying

$$\frac{M(x-z)}{2(x+R)}\geqslant\frac{M}{4}, \qquad x\geqslant x_0.$$

If $x_0\leqslant R$ define $\tau_2 := M/4$; otherwise put

$$\tau_2 := \min\left\{\frac{M}{4}, \min\left\{\frac{M(x-z)}{2(x+R)} \,|\, x\in[R, x_0]\right\}\right\}.$$

Note that in any case $\tau_2>0$ is satisfied since R was chosen to be greater than $|z|+1$.

Setting $\tau := \min\{\tau_1, \tau_2\}>0$ we define

$$p^*(x) := p(x)+\tau Q_R(x).$$

p^* is a polynomial of degree at most n satisfying

$$\begin{aligned} f(x)-p^*(x) &\geqslant f(x)-p(x)\geqslant 0, && x\in(-\infty, R]\backslash K,\\ f(x)-p^*(x) &\geqslant f(x)-p(x)-d\geqslant 0, && x\in K,\\ f(x)-p^*(x) &\geqslant f(x)-p(x)-\frac{M}{2}(x-x_1)^2\cdots(x-x_{(n-1)/2})^2(x-z) && \\ &= f(x)-p_1(x)\geqslant 0, && x\in(R,\infty), \end{aligned}$$

and

$$\int_{-\infty}^{\infty} p^*(t)\,d\mu(t) > \int_{-\infty}^{\infty} p(t)\,d\mu(t).$$

This gives the contradiction to $p\in BL_n(f)$. Therefore, $M=0$ and $p_1=p_2$.

(2.2) $-\infty=a$ *and* $b<\infty$. By means of Lemma 2 the nonnegative function $f-p$ has at least $(n+1)/2$ zeros in $(-\infty, b]$. If they are all located in $(-\infty, b)$ we immediately obtain $p_1=p_2$ by counting the multiple zeros of p_1-p_2. In the other case, however, the zeros of p_1-p_2 are $x_1<\cdots<x_{(n-1)/2}<x_{(n+1)/2}=b$ and, therefore, there exists a constant $M\in\mathbb{R}$ satisfying

$$p_1(x)-p_2(x)=M(x-x_1)^2\cdots(x-x_{(n-1)/2})^2(b-x).$$

Now, the proof runs in the same way as in (1).

(2.3) $-\infty<a$ *and* $b=\infty$. The proof is essentially the same as in (2.2). ∎

Remark. Let us note that it is in general not possible to obtain the above uniqueness result by using well-known transformation techniques (cf. [8, Chaps. V and VI]) and then applying the uniqueness theorem for one-sided L_1-approximation by differentiable T-systems on compact intervals (cf. [2, Theorem 3.3]) or its generalizations (cf. [13, Theorems 6, 7, and 8]). This strategy does not work since for fast growing f near $+\infty$ and/or $-\infty$ we cannot simultaneously guarantee that the transformed function f is continuous at the end points of the compact interval and that the transformed polynomials do not vanish identically at these points. If, however, we allow the transformed polynomials to vanish identically at the end points of the new compact interval they are no longer a T-system and even the remarkable Theorems 6 (in case of n even), 7, and 8 of [13] would not yield a positive answer concerning uniqueness.

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